Section 8.4

last few pages of the section

Section 8.4 (last pages)

Subfactors - factorgrous of subgroups $\neq$ subgroups of factor groups
- same thing

G - a group N - a normal subgroup of G

Th 8.21 $N \subset K \subset G$

$K/N \subset G/N$ - subgroup

Pf N is normal in G means

$aN = Na$ for every $a \in G$

Therefore $aN = N$ for every $a \in K$

Thus N is normal in K and

K/N makes sense, is a group

$K/N = \{Na \mid a \in K\}$

is a subset of

$G/N = \{Na \mid a \in G\}$

Th 8.24 For every subgroup $T \subset G/N$, there exists H, a subgroup of G s.t.

$N \subset H \subset G$ and $T \simeq H/N$

Pf $H = \{a \in G \mid Na \in T\}$

H is a subgroup in G - Exer 23 p 271

$a, b \in H$ check: $ab \in H$ ✓

$Na \in T, Nb \in T$ then $(Na)(Nb) = N(ab) \in T$ implies $ab \in T$

$a \in H$ check: $a^{-1} \in H$ ✓

$Na \in T$ thus $(Na)^{-1} \in T$, $(Na)^{-1} = Na^{-1} \in T$

$H \supset N$ If $a \in N$, then $Na = N$ is the identity in T

$H/N = \{a \in G \mid Na \in T\} / N = \{Na \mid Na \in T\}$

$\equiv \{Na \in T\} = T$

Th 8.22

Let the intermediate subgroup K

$$N \subset K \subset G$$

be normal in G.

Then $K/N \subset G/N$

is a normal subgroup and

$$(G/N)/(K/N) \simeq G/K$$ – the 3^d isomorphism theorem

Cor 8.23

$$N \subset K \subset G$$

K is normal in G <u>iff</u>

K/N is normal in G/N

<u>Def</u> A group G is called <u>simple</u> if the only normal subgroups it has are trivial: $\langle e \rangle$, G

<u>Prop</u> Let G be a finite group.

Let G_1 be <u>a</u> normal subgroup of the largest order possible.

$G_1 \neq G$

If there are several of the same order, G_1 may be any one of them.

Then <u>G/G_1 is simple.</u>

Pf. Assume, for a contradiction,

that $G/G_1 \supset M$, a normal subgroup

$G_1 \subsetneq T \subset G \qquad T/G_1 \simeq M$

T is normal in G and larger than G_1, which is a contradiction.

Procedure

Starting with $G = G_0$, a finite group, we construct G_1, after that $G_2 \dots$

$$G_0 \supsetneq G_1 \supsetneq G_2 \supsetneq \dots \supsetneq \langle e \rangle$$

All factors G_{i-1}/G_i are simple groups – composition factors for the group G

Th (Jordan-Hölder)

A group completely determines its composition factors.
Specifically, if one has a series $G_0 \supsetneq G_1 \supsetneq G_2 \supsetneq \dots \supsetneq \langle e \rangle$
such that all quotients G_{i-1}/G_i are simple,
the set of these simple quotients is determined uniquely,
independently on how the series is constructed.
This set depends on the initial finite group only.

Classification of simple groups is mostly done by 1980,
presumably last gaps in the proof (and results) were fixed ~ 2004.

There are 18 infinite families of simple groups

26 sporadic simple groups

A very elementary one (out of 18): The group $\mathbb{Z}_p$ (cyclic out of p elements) is simple for any prime p.